FORCED OSCILLATIONS OF A QUASILINEAR SYSTEM IN THE PRESENCE OF A RAPIDLY CHANGING EXTERNAL FORCE

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The author investigates the problem of the existence and properties of bounded oscillations of a quasilinear system of the saddle type in the presence of a rapidly changing external force*. The obtained results generalize a theorem of Farnell, Langenhop and Levinson.

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1. Linear system with the (α, β) property. Let us consider the linear homogeneous system of ordinary differential equations

$$\frac{dx}{dt} = P(t)x, \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{1.1}$$

Here x is a real n-dimensional vector (a column matrix) and $P(t)$ is a real $n \times n$ matrix which is defined and continuous for all values of t $(-\infty < t < +\infty)$.

Let $X(t)$ be a canonical fundamental matrix of the solutions of the system (1.1) such that $X(0) = I_n$, where I_n is the unit $n \times n$ matrix. Then

* For the sake of simplicity in the presentation, all considered matrices and functions will be assumed to be real. The basic results are also valid, with obvious modifications, for systems of differential equations whose right-hand sides are continuous complex functions of a real independent variable t, and analytic functions of the dependent variables; the solutions may be complex-valued.

$$X(t, t_0) = X(t) X^{-1}(t_0) \qquad (-\infty < t < +\infty, \ -\infty < t_0 < +\infty) \tag{1.2}$$

is a fundamental system of solutions determined by the initial condition: $X(t_0, t_0) = I_n$ (Cauchy's matrix).

Definition 1.1. We shall say that the system (1.1) (or the matrix $P(t)$) has the (α, β) property if its Cauchy matrix $X(t, t_0)$ can be broken up into two subsystems of solutions

$$X_\alpha(t, t_0) = X(t) A X^{-1}(t_0), \qquad X_\beta(t, t_0) = X(t) B X^{-1}(t_0) \tag{1.3}$$

such that

$$\|X_\alpha(t, t_0)\| \leqslant a e^{-\alpha(t-t_0)} \text{ when } t \geqslant t_0, \qquad \|X_\beta(t, t_0)\| \leqslant b e^{\beta(t-t_0)} \text{ when } t \leqslant t_0 \tag{1.4}$$

Here a, b, α, and β are positive numbers; A and B are constant $n \times n$ matrices, and $A + B = I_n$;

$$\|X\| = \sum_{i,j} |x_{i,j}| \text{ — norm of the matrix } X = [x_{ij}].$$

The inequalities (1.4) are generalizations of known Persidskii's conditions [2], which are obtained when $A = I_n$ and $B = 0$. Similar two-sided conditions were used by Maizel' [3]. Analogous conditions for systems of differential equations in Banach spaces are given by Krein [4], and Massera and Schäffer [5].

We note that the system (1.1) has the property (α, β) if the matrix $P(t) = P$ is constant, while $m (m \leqslant n)$ of its characteristic numbers $\lambda_1, \ldots, \lambda_m$ have negative real parts, and $n - m$ characteristic numbers have positive real parts. In this case

$$A = S \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} S^{-1}, \qquad B = S \begin{bmatrix} 0 & 0 \\ 0 & I_{n-m} \end{bmatrix} S^{-1}$$

Here I_m and I_{n-m} are unit matrices of order m and $n - m$, while S is a nonsingular matrix which transforms the matrix P to the Jordan form; m is the rank of the matrix $X(t, t_0)$, and $n - m$ is the rank of the matrix $X(t, t_0)$.

In the more general case, the property (α, β) occurs when the system (1.1) is reducible [6,7] to a system with a constant matrix whose characteristic numbers have real parts different from zero. In particular, for example, a linear homogeneous system with a periodic matrix $P(t)$, without zero or pure imaginary characteristic exponents, has the (α, β) property.

We note also that the system (1.1) will possess the (α, β) property if

$$P(t) = \Lambda(t) + Q(t)$$

where $\Lambda(t) = [\lambda_1(t), \ldots, \lambda_n(t)]$ is a diagonal matrix and $Q(t)$ is absolutely integrable on $(-\infty, +\infty)$ (two-sided L-diagonal system [8,9]), and

$$\lambda_1(t) \leqslant \ldots \leqslant \lambda_m(t) \leqslant -\alpha < 0 < \beta \leqslant \lambda_{m+1}(t) \leqslant \ldots \leqslant \lambda_n(t)$$

Lemma 1.1. Let the matrix $P(t)$ be continuous on $(-\infty, +\infty)$ and possess the property (α, β). Then there exists a bounded $n \times n$ matrix $G(t, t_1) \in C'$ for $t \neq t_1$ $(-\infty < t < +\infty;\ -\infty < t_1 < +\infty)$ such that

$$\begin{aligned}
&(1) \qquad G(t, t-0) - G(t, t+0) = I_n \\
&(2) \qquad G'_t(t, t_1) = P(t)\, G(t, t_1) \text{ when } t \neq t_1 \\
&\qquad\quad\ G_{t_1}'(t, t_1) = -G(t, t_1)\, P(t_1) \text{ when } t \neq t_1 \\
&(3) \qquad \| G(t, t_1) \| \leqslant c e^{-\gamma |t - t_1|} \quad (c,\ \gamma = \text{const} > 0)
\end{aligned} \tag{1.5}$$

(4) If, in addition, $f(t)$ is a vector function of the $n \times 1$ type, and continuous when $-\infty < t < +\infty$, then the nonhomogeneous system

$$\frac{dy}{dt} = P(t)\, y + f(t) \tag{1.6}$$

has a bounded solution on $-\infty < t < +\infty$

$$\eta(t) = \int_{-\infty}^{+\infty} G(t, t_1)\, f(t_1)\, dt_1 \tag{1.7}$$

provided that the integral (1.7) converges uniformly in t on every finite interval $(-l, l) \subset (-\infty, +\infty)$, and is bounded on $(-\infty, +\infty)$.

Proof. Let us set

$$G(t, t_1) = \begin{cases} X_\alpha(t, t_1) & \text{when } t > t_1 \\ -X_\beta(t, t_1) & \text{when } t < t_1 \end{cases}$$

where the matrices $X_\alpha(t, t_0)$ and $X_\beta(t, t_0)$ are given by Formulas (1.3).

Obviously, we have

1) $G(t,\ t-0) - G(t,\ t+0) = X(t)\,AX^{-1}(t) + X(t)\,BX^{-1}(t)$
$= X(t)\,(A+B)\,X^{-1}(t) = X(t)\,I_n X^{-1}(t) = I_n$

2) if $C = A$ when $t > t_1$ and $C = -B$ when $t < t_1$, then

$$G(t,\ t_1) = X(t)\,CX^{-1}(t_1) \tag{1.8}$$

and when $t \neq t_1$ we obtain

$$G_t'(t,\ t_1) = X'(t)\,CX^{-1}(t_1) = P(t)\,X(t)\,CX^{-1}(t_1) = P(t)\,G(t,\ t_1)$$

$$G_{t_1}'(t,\ t_1) = -X(t)\,CX^{-1}(t_1)\,X'(t_1)\,X^{-1}(t_1) =$$
$$= -X(t)\,CX^{-1}(t_1)\,P(t_1)X(t_1)\,X^{-1}(t_1) = -G(t,\ t_1)\,P(t_1)$$

3) Making use of the inequalities (1.4), we find that

$$\|G(t,t_1)\| \leqslant ae^{-\alpha(t-t_1)} \text{ when } t>t_1, \qquad \|G(t,t_1)\| \leqslant be^{\beta(t-t_1)} \text{ when } t<t_1,$$

Hence, setting $c = \max(a,\ b)$ and $\gamma = \min(\alpha,\ \beta)$, we obtain (1.5).

4) Let us consider the vector function determined by Formula (1.7). We have

$$\eta(t) = \int_{-\infty}^{t} G(t,\ t_1)\,f(t_1)\,dt_1 + \int_{t}^{+\infty} G(t,\ t_1)\,f(t_1)\,dt_1$$

Performing the formal differentiation with respect to t of the last expression, and taking into account 1 and 2, we obtain

$$\eta'(t) = G(t,\ t-0)\,f(t) + \int_{-\infty}^{t} P(t)\,G(t,\ t_1)\,f(t_1)\,dt_1 - G(t,\ t+0)\,f(t) +$$
$$+ \int_{t}^{+\infty} P(t)\,G(t,\ t_1)\,f(t_1)\,dt_1 = f(t) + P(t)\int_{-\infty}^{+\infty} G(t,\ t_1)\,f(t_1)\,dt_1 \tag{1.9}$$

Since the matrix $P(t)$ is bounded on every finite interval $(-l,\ l)$, and since we assume that the integrals

$$\int_{-\infty}^{t} G(t,t_1)\,f(t_1)\,dt_1, \qquad \int_{t}^{+\infty} G(t,\ t_1)\,f(t_1)\,dt_1$$

converge uniformly on $(-l,\ l)$, it is obvious that the integrals

$$\int_{-\infty}^{t} G_t'(t,\ t_1)\,f(t_1)\,dt_1, \qquad \int_{t}^{+\infty} G_t'(t,t_1)\,f(t_1)\,dt_1$$

also converge uniformly on $(-l, l)$. Hence, by a known theorem of analysis, the derivative vector function $\eta(t)$ is determined by (1.9) on $-\infty < t < +\infty$. From this it follows that

$$\eta'(t) = f(t) + P(t)\eta(t)$$

i.e. $\eta(t)$ is a bounded solution of the nonhomogeneous system (1.6).

Corollary. The bounded solution $\eta(t)$ of the homogeneous system (1.6) exists if: (a) the vector function $f(t)$ is continuous and bounded on $(-\infty, +\infty)$; or (b) the matrix $P(t)$ and the vector function

$$F(t) = \int_0^t f(t_1)\, dt_1$$

are continuous and bounded on $(-\infty, +\infty)$ (the boundedness of the vector function $f(t)$ is not assumed here).

Proof. a) If $\| f(t) \| \leqslant k$ when $t \in (-\infty, +\infty)$, where k is a positive constant, then the uniform convergence of the integral (1.7) on any finite interval, and its boundedness are obvious.

b) Let $\| P(t) \| \leqslant c_1$, and $\| F(t) \| \leqslant c_2$, where c_1 and c_2 are positive constants. Then integration by parts yields

$$\begin{aligned}\eta(t) &= \int_{-\infty}^{+\infty} G(t, t_1) f(t_1)\, dt_1 = \int_{-\infty}^{+\infty} G(t, t_1) F'(t_1)\, dt_1 \\ &= G(t, t_1) F(t_1) \Big|_{t_1=-\infty}^{t_1=+\infty} - \int_{-\infty}^{+\infty} G_{t_1}'(t, t_1) F(t_1)\, dt_1 \\ &= \int_{-\infty}^{+\infty} G(t, t_1) P(t_1) F(t_1)\, dt_1 = \int_{-\infty}^{+\infty} G_1(t, t_1) F(t_1)\, dt_1 \qquad (1.10)\end{aligned}$$

Here

$$G_1(t, t_1) = G(t, t_1) P(t_1), \qquad \| G_1(t, t_1) \| \leqslant \| G(t, t_1) \| \, \| P(t_1) \| \leqslant cc_1 e^{-\gamma|t-t_1|}$$

Hence, the integral (1.10) converges uniformly in t on every finite interval of the axis $-\infty < t < +\infty$.

Note 1.1. If $f(t)$ is bounded, then

$$\| \eta(t) \| \leqslant \Gamma \sup_t \| f(t) \| \qquad \left(\Gamma = \sup_t \int_{-\infty}^{+\infty} \| G(t, t_1) \| dt_1 \leqslant \frac{2c}{\gamma} \right)$$

1.2. If $P(t)$ and $F(t)$ are bounded, then in view of Formula (1.10) we have

$$\|\eta(t)\| \leqslant \Gamma_1 \sup_t \|F(t)\| \qquad \left(\Gamma_1 = \sup_t \int_{-\infty}^{+\infty} \|G_1(t, t_1)\| \, dt_1\right)$$

Lemma 1.2. If the matrix $P(t)$ and the vector function $f(t)$ are almost periodic, and if $P(t)$ has the (a, β) property while the homogeneous system (1.1) has no nontrivial solution bounded on the axis $-\infty < t < +\infty$, then the bounded solution $\eta(t)$ of the nonhomogeneous system (1.6) is also almost periodic.

This lemma is a slight modification of a theorem by Favard [16].

Proof. Let τ be a general "almost period" of the matrix $P(t)$ and of $f(t)$ with an accuracy of ϵ, i.e.

$$\begin{aligned} \|\Delta_\tau P(t)\| &= \|P(t+\tau) - P(t)\| < \varepsilon \\ \|\Delta_\tau f(t)\| &= \|f(t+\tau) - f(t)\| < \varepsilon \end{aligned} \qquad \text{when } -\infty < t < +\infty$$

Setting $\Delta_\tau \eta(t) = \eta(t+\tau) - \eta(t)$, we obtain

$$\frac{d}{dt}[\Delta_\tau \eta(t)] = P(t) \cdot \Delta_\tau \eta(t) + [\Delta_\tau P(t)\, \eta(t+\tau) + \Delta_\tau (t)]$$

Since the matrix $\Delta_\tau P(t) \cdot \eta(t) + \Delta_\tau f(t)$ is bounded, and since by hypothesis of Lemma 1.2 the nonhomogeneous system with matrix $P(t)$, and with a bounded free term, has only one bounded solution, it follows that

$$\Delta_\tau \eta(t) = \int_{-\infty}^{+\infty} G(t, t_1)\, [\Delta_\tau P(t_1) \cdot \eta(t_1+\tau) + \Delta_\tau f(t_1)]\, dt_1$$

Therefore

$$\|\Delta_\tau \eta(t)\| \leqslant \Gamma\varepsilon(M+1) \qquad \text{for } -\infty < t < +\infty \qquad (M = \sup_t \|\eta(t)\| < +\infty)$$

and, hence, $\eta(t)$ is an almost periodic vector function.

Note. If in the condition (1.4) $A = I_n$ and $B = 0$, or $A = 0$ and $B = I_n$, then the homogeneous system has no nontrivial solution which is bounded on $(-\infty, +\infty)$.

Indeed, if, for example, $A = I_n$ and $B = 0$, then $X(t, t_0) = X_\alpha(t_1, t_0)$ and the correctness of the remark follows easily from the inequality of (1.4).

2. Existence of bounded solutions of a quasilinear system.

Let

$$\frac{dy}{dt} \doteq P(t)\, y + f(\omega t, y, \mu) + e(\omega t) \tag{2.1}$$

where y is the $n \times 1$ solution vector; μ is a small real parameter, ω is a large positive parameter; $P(t)$ is an $n \times n$ continuous matrix bounded on $(-\infty, +\infty)$ and possessing the (α, β) property; $f(t, y, \mu)$ is an $n \times 1$ vector function defined and continuous in the region $D = \{ |t| < +\infty, \|y\| < +\infty, |\mu| < \Lambda \}$, and satisfying in every subregion $D_r = \{ |t| < +\infty, \|y\| \leqslant r < +\infty, |\mu| \leqslant \mu_0 < \Lambda \}$ the Lipschitz condition

$$\| f(t, y, \mu) - f(t, z, \mu) \| \leqslant L(r, \mu_0) \| y - z \| \qquad (\|y\| \leqslant r, \ \|z\| \leqslant r) \tag{2.2}$$

where $L(r, \mu_0)$ is a positive scalar function independent of t, whereby

$$\lim L(r, \mu_0) = 0 \qquad \text{as} \quad \mu_0 \to 0 \tag{2.3}$$

and furthermore

$$\| f(t, 0, \mu) \| \leqslant k \qquad (k = \text{const}) \tag{2.4}$$

Finally, it is assumed in (2.1) that the continuous $n \times 1$ vector function $e(t)$ has a bounded integral on $(-\infty, +\infty)$

$$E(t) = \int_0^t e(t_1)\, dt_1, \qquad \| E(t) \| \leqslant k_1 \qquad (k_1 = \text{const}) \tag{2.5}$$

Theorem 2.1. There exists a positive constant μ_0 such that when $|\mu| \leqslant \mu_0$ the system (2.1) has at least one solution $\eta = \eta(t)$ bounded on the entire axis $-\infty < t < +\infty$. (If $e(t)$ is bounded, then the boundedness of $E(t)$ is not required.)

This result is analogous to a theorem of Perron [11]; the assumptions relative to the real part of the system (2.1) are, however, more general; in particular, we do not assume the boundedness with respect to t.

The author has obtained an analogous result for the case of a constant matrix, $P(t) = \text{const}$ [12].

Proof. Let us consider the singular integral equation

$$y(t) = \int_{-\infty}^{+\infty} G(t, t_1)\, [f(\omega t_1, y(t_1), \mu) + e(\omega t_1)]\, dt_1 \tag{2.6}$$

where $G(t, t_1)$ is determined on the basis of Lemma 1.1, and satisfies the inequality (1.5). Because of Lemma 1.1, the continuous bounded solution $\eta(t)$ of the integral equation (2.6) is a bounded solution of the system (2.1).

For the proof of the existence of the solution $\eta(t)$ we make use of the method of successive approximations setting

$$y^{(0)}(t) = 0$$

$$y^{(p)}(t) = \int_{-\infty}^{+\infty} G(t, t_1)\,[f(\omega t_1, y^{(p-1)}(t_1), \mu) + e(\omega t_1)]\,dt_1 \quad (p = 1, 2, \ldots) \tag{2.7}$$

Taking into account the boundedness of the function $E(t)$, we have

$$y^{(1)}(t) = \int_{-\infty}^{+\infty} G(t, t_1)\, f(\omega t_1, 0, \mu)\,dt_1 + \int_{-\infty}^{+\infty} G(t, t_1)\, e(\omega t_1)\,dt_1$$

$$= \int_{-\infty}^{+\infty} G(t, t_1)\, f(\omega t_1, 0, \mu)\,dt_1 + \frac{1}{\omega}\int_{-\infty}^{+\infty} G(t, t_1)\frac{d}{dt_1}E(\omega t_1)\,dt_1$$

$$= \int_{-\infty}^{+\infty} G(t, t_1)\, f(\omega t_1, 0, \mu)\,dt_1 + \frac{1}{\omega}\, G(t, t_1)\, E(\omega t_1)\Big|_{t_1=-\infty}^{t_1=+\infty} -$$

$$- \frac{1}{\omega}\int_{-\infty}^{+\infty} G_{t_1}'(t, t_1)\, E(\omega t_1)\,dt_1 = \int_{-\infty}^{+\infty} G(t, t_1)\, f(\omega t_1, 0, \mu)\,dt_1 +$$

$$+ \frac{1}{\omega}\int_{-\infty}^{+\infty} G(t, t_1)\, P(t_1)\, E(\omega t_1)\,dt_1$$

From this it follows that on the basis of (2.4) and (2.5) we obtain

$$\|y^{(1)}(t)\| \leqslant \Gamma k + \frac{1}{\omega}\cdot \Gamma_1 k_1 = R \tag{2.8}$$

where

$$\Gamma = \sup_t \int_{-\infty}^{+\infty} \|G(t, t_1)\|\,dt_1, \qquad \Gamma_1 = \sup_t \int_{-\infty}^{+\infty} \|G(t, t_1)\, P(t_1)\|\,dt_1$$

Making use of the condition (2.3), let us select a positive number μ_0 so small that for $|\mu| < \mu_0$ the following inequality will hold:

$$L = L(2R, \mu_0) < \frac{1}{2\Gamma}$$

By mathematical induction one can easily prove that all approximations $y^{(p)}(t)$ $(p = 1, 2, \ldots)$ satisfy the inequality

$$\|y^{(p)}(t) - y^{(p-1)}(t)\| \leqslant \left(\frac{1}{2}\right)^{p-1} R$$

when $-\infty < t < +\infty$, and hence that

$$\|y^{(p)}(t)\| \leqslant \sum_{q=1}^{p} \|y^{(q)}(t) - y^{(q-1)}(t)\| \leqslant \left(1 + \frac{1}{2} + \ldots + \frac{1}{2^{p-1}}\right) R < 2R$$

By the usual procedure one proves the existence of

$$\eta(t) = \lim y^{(p)}(t) \quad \text{where } p \to \infty \tag{2.9}$$

for which one has the estimate

$$\|\eta(t) - y^{(p)}(t)\| \leqslant \left(\frac{1}{2}\right)^{p-1} R, \qquad \|\eta(t)\| \leqslant 2R \tag{2.10}$$

where R is given by (2.8).

From the last inequality it follows that

$$y^{(p)}(t) \rightrightarrows \eta(t) \qquad (-\infty < t < +\infty)$$

The limit function $\eta(t)$ will be the unique* bounded continuous solution of the integral equation (2.6), and, hence, also a bounded solution of the quasilinear system (2.1).

Note. The bounded solution $\eta(t)$ of the nonlinear system (2.1) satisfying the integral equation (2.6) will be called a regular solution.

Corollary. If $f(t, 0, \mu) \equiv 0$, then the amplitude of the regular bounded solution $\eta(t)$ will be arbitrarily small provided that ω is sufficiently large.

Indeed, setting $k = 0$, and taking account of (2.8) and (2.10), we have

$$\|\eta(t)\| \leqslant \frac{2k_1\Gamma_1}{\omega} \tag{2.11}$$

Theorem 2.2. Suppose that the quasilinear system (2.1) is such that the matrix $P(t)$ and also the vector functions $f(t, y, \mu)$ and $e(t)$ are almost periodic functions of t, and that in addition the homogeneous system (1.1) with matrix $P(t)$ has no bounded nontrivial solutions on $-\infty < t < +\infty$. Then, if $|\mu| \leqslant \mu_0$, the bounded solution $\eta(t)$ of the quasilinear system (2.1) will also be almost periodic.

Biriuk [13], the author [12], and Langenhop [14] have proved

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* The uniqueness of the bounded solution $\eta(t)$ of the integral equation (2.6) is easily established by contradiction.

analogous theorems for the case of a constant matrix, $P(t)$ = const.

Proof. In view of Lemma 1.2, all successive approximations $y^{(p)}(t)$ ($p = 1, 2, \ldots$) of the bounded solution (t) are almost periodic. Since

$$y^{(p)}(t) \rightrightarrows \eta(t) \qquad \text{on } (-\infty, +\infty)$$

the function $\eta(t)$ will also be an almost-periodic vector function.

Note. If $A = I_n$ and $B = 0$, or if $A = 0$ and $B = I_n$, the requirement of the absence of a nontrivial bounded solution of the homogeneous system (1.1) becomes superfluous (see Section 1, Note on Lemma 1.2).

Corollary. If the matrix $P(t)$ is periodic of period T/ω, and the functions $f(t, y, \mu)$, $e(t)$ are periodic in t with the common period T, then under the hypotheses of Theorem 2.2, the bounded solution $\eta(t)$ of the quasilinear system (2.1) will be periodic of period T/ω.

If in addition $f(t, 0, \mu) \equiv 0$, then

$$\|\eta(t)\| \leqslant \frac{2\Gamma_1}{\omega} \sup \left\| \int_0^t e(t_1)\, dt_1 \right\| \qquad (0 \leqslant t \leqslant T) \tag{2.12}$$

Indeed, setting

$$z(t) = \eta\left(t + \frac{T}{\omega}\right) - \eta(t)$$

and taking into account the periodicity of the function $f(t, y, \mu)$, we obtain

$$z(t) = \int_{-\infty}^{+\infty} G(t, t_1)\left[f\left(\omega t_1, \eta\left(t_1 + \frac{T}{\omega}\right), \mu\right) - f(\omega t_1, \eta(t_1), \mu) \right] dt_1$$

From this it follows that

$$\sup_t \|z(t)\| \leqslant \int_{-\infty}^{+\infty} \|G(t, t_1)\| L \left\| \eta\left(t_1 + \frac{T}{\omega}\right) - \eta(t_1) \right\| dt_1 \leqslant$$
$$\leqslant \Gamma \frac{1}{2\Gamma} \sup_t \left\| \eta\left(t_1 + \frac{T}{\omega}\right) - \eta(t_1) \right\| = \frac{1}{2} \sup_t \|z(t)\|$$

Hence

$$\sup_t \|z(t)\| = \sup_t \left\| \eta\left(t + \frac{T}{\omega}\right) - \eta(t) \right\| = 0$$

i.e.

$$\eta\left(t + \frac{T}{\omega}\right) \equiv \eta(t) \qquad \text{for } -\infty < t < +\infty$$

This proves that the bounded solution $\eta(t)$ is periodic with period T/ω.

In regard to the inequality (2.12), it can be said that it is a direct consequence of Formula (2.11).

Note. In [1] there was considered a system of the type (2.1) (without the parameter μ) under the hypotheses: 1) the matrix $P(t)$ = const; 2) the functions $f(t, y)$, $e(t) = ke_0(t)$ $(k > 0)$, and $E(t)$ are periodic in t with period T; 3) $\| f(t, y) - f(t, z) \| \leqslant L \| y - z \|$, where L is a small enough positive constant; 4) $f(t, 0) \equiv 0$. For $\omega \geqslant \omega_0$ there is guaranteed the existence of a periodic solution $y = p(t)$ of period $1/\omega$, and such that

$$\| p(t) \| \leqslant \frac{k\rho}{1+\omega}$$

where ρ is a positive constant independent of the parameter k. Our theorem 2.2 strengthens this result.

3. Stability of the regular bounded solution of the quasilinear system. *Theorem 3.1.* Let the matrix $P(t)$ possess the (α, β) property and be such that the rank of $X_\alpha(t_0, t_0)$ is equal to $m (m \leqslant n)$, where t_0 is fixed. Then, if $|\mu| \leqslant \mu_0$, the regular bounded solution $\eta(t)$ of the quasilinear system (2.1) is asymptotically conditionally stable in the Liapunov sense when $t \to +\infty$ on the manifold S_m^+ of the solutions $y(t)$, which depends on the parameter m, and this stability is of the exponential type; namely, if

$$\|\eta(t)\| \leqslant 2R, \qquad y(t) \in S_m^+, \qquad \|y(t_0) - \eta(t_0)\| \leqslant \rho(t_0)$$

where $\rho(t_0)$ is a sufficiently small positive constant, then

$$\|y(t) - \eta(t)\| \leqslant \frac{R}{\rho(t_0)} \|y(t_0) - \eta(t_0)\| e^{-1/2\gamma(t-t_0)} \quad \text{for } t \geqslant t_0 \tag{3.1}$$

where γ is determined by Formula (1.5).

Note. Analogous results were obtained for the case $P(t)$ = const by Petrovskii [15], Coddington and Levinson [16], and by the author [12].

Proof. Let us set $\Delta \equiv \Delta(t) = y(t) - \eta(t)$. From the system (2.1) we have

$$\frac{d\Delta}{dt} = P(t)\Delta(t) + \varphi(t, \Delta(t)) \tag{3.2}$$

where

$$\varphi(t, \Delta(t)) = f(\omega t, \eta(t) + \Delta(t), \mu) - f(\omega t, \eta(t), \mu) \tag{3.3}$$

Let us consider the singular integral equation

$$\Delta(t) = X_\alpha(t, t_0)\, g + \int_{t_0}^{+\infty} G(t, t_1)\, \varphi(t_1, \Delta(t_1))\, dt_1 \tag{3.4}$$

where g is a constant $n \times 1$ vector. If $\Delta(t)$ is a continuous vector function satisfying Equation (3.4), then by differentiating Equation (3.4) with respect to t and taking into account the properties of the matrix $X_\alpha(t, t_0)$ and $G(t, t_1)$ when $t > t_0$, we obtain

$$\Delta'(t) = P(t)\, X_\alpha(t, t_0)\, g + [G(t, t-0) - G(t, t+0)]\, \varphi(t, \Delta(t)) + \\ + P(t) \int_{t_0}^{+\infty} G(t, t_1)\, \varphi(t_1, \Delta(t_1))\, dt_1 \equiv P(t)\, \Delta(t) + \varphi(t, \Delta(t))$$

Then the solution $\Delta(t)$ of the integral equation (3.4) will also be a solution of the differential equation (3.2).

Let us choose a positive number μ_1 so small that for $|\mu| \leqslant \mu_1$ the following inequality is true:

$$L_1 = L(3R, \mu_1) \leqslant \min\left(\frac{1}{2\Gamma}, \frac{\gamma q}{3c}\right) \qquad (0 < q < 1) \tag{3.5}$$

Next, we subject the norm of the vector g to the inequality

$$\|g\| \leqslant \frac{R_1}{a} \qquad (R_1 \leqslant R(1-q)) \tag{3.6}$$

and let

$$r = a\, \|g\| \tag{3.7}$$

For the purpose of finding the solution of the integral equation (3.4) in the class of functions $\{\|\Delta(t)\| \leqslant R$ when $t_0 \leqslant t < +\infty\}$ under the conditions (3.5) and (3.6), we make use of the method of successive approximations, and set

$$\Delta^{(0)}(t) = X_\alpha(t, t_0)\, g$$

$$\Delta^{(s)}(t) = \Delta^{(0)}(t) + \int_{t_0}^{+\infty} G(t, t_1)\, \varphi(t_1, \Delta^{(s-1)}(t_1))\, dt_1 \tag{3.8}$$

where $t \geqslant t_0$ $(s = 1, 2, \ldots)$. Since

$$\|\Delta^{(0)}(t)\| \leqslant a\, \|g\|\, e^{-\gamma(t-t_0)} \leqslant r e^{-1/2\gamma(t-t_0)}$$

$$\|\varphi(t, \Delta^{(0)}(t)\| \leqslant L_1 \|\Delta^{(0)}(t)\| \leqslant \frac{\gamma q}{3c} r e^{-1/2\gamma(t-t_0)}$$

we obtain by the use of the properties of the matrix $G(t, t_1)$ and Formulas (3.8) and (3.9)

$$\|\Delta^{(1)}(t)\| \leqslant \|\Delta^{(0)}(t)\| + \int_{t_0}^{+\infty} ce^{-\gamma|t-t_1|}\frac{\gamma q}{3c} re^{-1/2\gamma(t_1-t_0)}\,dt_1 \leqslant re^{-1/2\gamma(t-t_0)} + \frac{\gamma q}{3} rJ(t)$$

where

$$J(t) = \int_{t_0}^{+\infty} e^{-\gamma|t-t_1|}e^{-1/2\gamma(t_1-t_0)}dt_1 < \frac{3}{\gamma}e^{-1/2\gamma(t-t_0)}$$

Therefore

$$\|\Delta^{(1)}(t)\| < r\,(1+q)\,e^{-1/2\gamma(t-t_0)}$$

In general, if

$$\|\Delta^{(s-1)}(t)\| < r\,(1+q+\ldots+q^{s-1})\,e^{-1/2\gamma(t-t_0)} \tag{3.9}$$

then Formulas (3.8) and (3.5) yield

$$\|\Delta^{(s)}(t)\| \leqslant \|\Delta^{(0)}(t)\| + \int_{t_0}^{+\infty} ce^{-\gamma|t-t_1|}\frac{\gamma q}{3c} r\,(1+q+\ldots+q^{s-1})\,e^{-1/2\gamma(t_1-t_0)}dt_1 \leqslant$$

$$\leqslant re^{-1/2\gamma(t-t_0)} + \frac{\gamma r}{3}(q+q^2+\ldots+q^s)\,J(t) <$$

$$< r\,(1+q+\ldots+q^s)\,e^{-1/2\gamma(t-t_0)}$$

Hence, for any natural number we have the inequality

$$\|\Delta^{(s)}(t)\| < \frac{r}{1-q}e^{-1/2\gamma(t-t_0)} \leqslant Re^{-1/2\gamma(t-t_0)} \tag{3.10}$$

Furthermore, since $\|\eta(t) + \Delta^{(s)}(t)\| \leqslant 3R$, we have

$$\sup_t \|\Delta^{(s+1)}(t) - \Delta^{(s)}(t)\| \leqslant \sup_t \int_{t_0}^{+\infty} \|G(t, t_1)\| \times$$

$$\times \|f(\omega t_1, \eta(t_1) + \Delta^{(s)}(t_1)\ \mu) - f(\omega t_1, \eta(t_1) + \Delta^{(s-1)}(t_1), \mu)\|\,dt_1 \leqslant$$

$$\leqslant \frac{1}{2}\sup_t \|\Delta^{(s)}(t) - \Delta^{(s-1)}(t)\| \quad (s = 1, 2, \ldots)$$

Whence, just as in the proof of Theorem 2.1, we find that

$$\Delta^{(s)} \rightrightarrows \Delta(t) \text{ on } (-\infty, +\infty)$$

The limit vector function $\Delta(t)$ satisfies the integral equation (3.4), and hence also the differential equation (3.2). Since $\eta(t)$ satisfies Equation (2.1), it follows from the type of the equation (3.2) that $y(t) = \eta(t) + \Delta(t)$ will also be a solution of (2.1).

Taking the limit as $s \to \infty$ in (3.10), we obtain

$$\| \Delta (t) \| = \| y (t) - \eta (t) \| \leqslant \frac{r}{1-q} e^{-1/2\gamma(t-t_0)} = \frac{a}{1-q} \| g \| e^{-1/2\gamma(t-t_0)} \text{ for } t \geqslant t_0 \quad (3.11)$$

In view of the facts that the rank of the matrix $X_\alpha(t, t_0)$ is equal to m, and that $\phi(t, \Delta(t))$ is arbitrarily small in norm if the quantity $\| g \|$ is sufficiently small, it follows from the implicit function theory that for $\| g \| \leqslant g_0$ there exists a manifold $S_m^{+(0)}$ of initial values $\Delta(t_0)$, and a corresponding manifold of vector functions g depending on m parameters $h_1, \ldots, h_m$ such that

$$\| g \| \leqslant K (t_0) \| \Delta (t_0) \|$$

Here $K(t_0)$ is some positive function. The set of solutions $y(t) = \eta(t) + \Delta(t)$, where $\Delta(t_0) \in S_m^{+(0)}$, we shall denote by S_m^+. From Formulas (3.11) with $y(t) \in S_m^+$, we obtain

$$\| y (t) - \eta (t) \| \leqslant \frac{a}{1-q} K (t_0) \| y (t_0) - \eta (t_0) \| e^{-1/2\gamma(t-t_0)} \qquad \text{when } t \geqslant t_0$$

provided only that

$$\| y (t_0) - \eta (t_0) \| \leqslant \frac{R}{a_1 K (t_0)} = \rho (t_0), \qquad a_1 = \max\left(\frac{a}{1-q}, \frac{R}{g_0}\right)$$

Hence

$$\| y (t) - \eta (t) \| \leqslant \frac{R}{\rho (t_0)} \| y (t_0) - \eta (t_0) \| e^{-1/2\gamma(t-t_0)} \qquad (3.12)$$

Corollary 3.1. If p is the rank of the matrix $X_\beta(t_0, t_0)$, then the regular bounded solution $\eta(t)$ of the quasilinear system (2.1) is conditionally asymptotically stable as $t \to -\infty$ on the manifold S_p^- of solutions depending on a parameter p; furthermore, this stability is of the exponential type.

Corollary 3.2. If the rank m of the matrix $X_\alpha(t_0, t_0)$ is equal to the order of the system (1.1), then the regular bounded solution $\eta(t)$ of the quasilinear system (2.1) is exponentially stable in the Liapunov sense when $t \to +\infty$ (compare [1]).

BIBLIOGRAPHY

1. Farnell, A.B., Langenhop, C.E. and Levinson, N., Forced periodic solutions of a stable nonlinear system of differential equations. *Journ. of Math. and Phys.* 29, No. 1, 1950.

2. Persidskii, K.P., K teorii ustoichivosti integralov sistem differentsial'nykh uravnenii (Theory of the stability of the integrals of systems of differential equations). *Izv. Fiz.-matem. ob-va pri Kazanskom gos. un-te* Vol. 8, 1936-1937.

3. Maizel', A.D., Ob ustoichivosti reshenii sistem differentsial'nykh uravnenii (On the stability of the solutions of systems of differential equations). *Tr. Ural'sk. politekhn. in-ta* No. 5, Matematika, 1954.

4. Krein, M.G., O nekotorykh voprosakh sviazannykh s krugom idei Liapunova v teorii ustoichivosti (On some questions connected with the ideas of Liapunov's theory of stability). *Usp. matem. nauk* Vol. 3, 3(25), 1948.

5. Massera, J.L. and Schäffer, J.J., Linear differential equations and functional analysis, I. *Ann. Math.* 67, No. 3, 1958.

6. Liapunov, A.M., *Obshchaia zadacha ob ustoichivosti dvizheniia (General Problem on the Stability of Motion)*. ONTI, 1935.

7. Erugin, N.P., Privodimye sistemy (Reducible systems). *Tr. Matem. in-ta im V.A. Steklova* Vol. 13, 1946.

8. Levinson, N., The asymptotic nature of solutions of linear systems of differential equations. *Duke Math. J.* Vol. 15, 1948.

9. Rappoport, H.M., *O nekotorykh asimptoticheskikh metodakh v teorii differentsial'nykh uravnenii (On Some Asymptotic Methods in the Theory of Differential Equations)*. Kiev, Izd-vo AN USSR, 1954.

10. Favard, J., Sur les équations différentielles à coefficients presque périodiques. *Acta Math.* 51, 31-81, 1927.

11. Perron, O., Die Stabilitätsfrage bei Differentialgleichungen. *Math. Zeit.* 32, 703-728, 1930.

12. Demidovich, B.P., Ob ogranichennykh resheniiakh nekotoroi nelineinoi sistemy obyknovennykh differentsial'nykh uravnenii (On bounded solutions of a nonlinear system of ordinary differential equations). *Matem. sb.* Vol. 40 (82), 1, 1956.

13. Biriuk, G.I., Ob odnoi teoreme sushchestvovaniia pochti periodicheskikh reshenii nekotorykh sistem nelineinykh differentsial'nykh uravnenii s malym parametrom (On a theorem on the existence of almost-periodic solutions of some systems of nonlinear differential equations with a small parameter). *Dokl. Akad. Nauk SSSR* Vol. 96, No. 1, 1954.

14. Langenhop, C.E., Note on almost-periodic solutions of nonlinear differential equations. *J. Math. and Phys.* 38, No. 2, 1959.

15. Petrovskii, I.G., O povedenii integral'nykh krivykh sistemy obyknovennykh differentsial'nykh uravnenii vblizi osoboi tochki (On the behavior of integral curves of a system of ordinary differential equations in the neighborhood of a singular point). *Matem. sb.* 41 : 1, 1934.

16. Coddington, E.A. and Levinson, N., *Teoriia obyknovennykh differentsial'nykh uravnenii (Theory of Ordinary Differential Equations)*, Chapt. 13. IIL, 1958. (Original edition, McGraw-Hill, 1955.)

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